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Best Quadrature Formulas and Splines*

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In this paper best quadrature formulas in the sense of Sard with fixed knots corresponding to splines satisfying boundary conditions are characterized. The motivation and inspiration stem from the work of Schoenberg [11–13], which we generalize and refine. Apart from their independent interest, these extensions are essential for our efforts in determining "optimal" quadrature formulas. ("Optimal", as distinguished from "best," allows the knots, in addition to the coefficients of the quadrature expression, to be regarded as free variables.)

The usual expression of a quadrature formula for a linear functional $L(f)$ defined for continuous functions on [0, 1] with preassigned knots $\{\xi_k\}_1^r$, $0 < \xi_1 < \cdots < \xi_r < 1$, is

$$Q(f) = \sum_{k=1}^{r} c_k f(\xi_k).$$

Each specification of $\{c_k\}$ provides a quadrature formula. The linear functional

$$R(f) = L(f) - Q(f)$$

is called the *remainder functional.* A quadrature formula $Q^*(f)$ with remainder $R^*(f)$ is said to be *best in the sense of Sard*, if $\{c_k^*\}$ corresponding to $Q^*(f)$ is determined in such a manner that

$$R^*(f) = 0 \qquad \text{when } f \text{ is a polynomial of degree } \leqslant n - 1^1 \tag{0.1}$$

and

$$\inf_{\{c_k\}} \sup_{\substack{f \in C^n[0,1] \\ \|f\|_n \leqslant 1}} | R(f)| = \sup_{\|f\|_n \leqslant 1} | R^*(f)|, \tag{0.2}$$

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[1] With a proper formulation (see Section 7) we will see that (0.1) is mostly superfluous for the objective of (0.2).

where $\|f\|_n^2 = \int_0^1 |f^{(n)}(x)|^2 \, dx$, and the quadrature formulas competing under the infinum sign are those satisfying (0.1).

We concentrate principally on determining the quadrature formula best in the sense of Sard for the special functional $L(f) = \int_0^1 f(x) \, dx$. Some extensions are dealt with in Section 6.

In dealing with these problems and their extensions to encompass the case of free knots it is useful and of independent interest to broaden the concept of quadrature formulas to include boundary terms. Specifically, we will consider quadrature formulas for $L(f)$, involving specific derivatives of f at the boundary points, of the type

$$Q(f) = \sum_{k=1}^{r} c_k f(\xi_k) + \sum_{\mu=1}^{n-p} A_\mu f^{(i_\mu')}(0) + \sum_{v=1}^{n-q} B_v f^{(j_v')}(1), \tag{0.3}$$

where $\{i_\mu'\}$ and $\{j_v'\}$ are prescribed and

$$0 \leqslant i_1' < i_2' < \cdots < i_{n-p}' \leqslant n-1, \quad 0 \leqslant j_1' < j_2' < \cdots < j_{n-q}' \leqslant n-1.$$

More generally, we will be concerned with quadrature formulas of the type

$$Q(f) = \sum_{k=1}^{r} c_k f(\xi_k) + \sum_{i=p+1}^{n} A_i \tilde{U}_i(f) + \sum_{i=q+1}^{n} B_i \tilde{V}_i(f), \tag{0.4}$$

where c_k, A_i and B_i are free real constants and $\tilde{U}_\alpha$, $\tilde{V}_\gamma$ are special linear forms, viz.,

$$\begin{aligned} \tilde{U}_\alpha(f) &= \sum_{\beta=0}^{n-1} \tilde{A}_{\alpha\beta} f^{(\beta)}(0), \qquad \alpha = p+1, \ldots, n, \\ \tilde{V}_\gamma(f) &= \sum_{\beta=0}^{n-1} \tilde{B}_{\gamma\beta} f^{(\beta)}(1), \qquad \gamma = q+1, \ldots, n. \end{aligned} \tag{0.5}$$

In other words, the quadrature part involves point evaluations of f at ξ_k and certain linear combinations of the derivatives of f (up to order $m-1$) at 0 and 1.

The formulation and solution of the problem of ascertaining the best quadrature formulas of the types (0.3) or (0.4) is elaborated in Sections 2 and 3, respectively. Some important examples of (0.5) are highlighted in Section 4.

Section 5 treats a case of quadrature formulas related to periodic boundary conditions. In Section 6, we indicate a series of extensions of the preceding results including the modifications needed for securing the best quadrature formulas for linear functionals of the from $L(f) = \int_0^1 f(x)\, w(x)\, dx$, where $w(x)$ is a continuous positive weight function.

A general perspective on and a framework for securing the best and optimal quadrature formulas is set forth in Section 7. Here, connections to problems of best approximation in the complex domain are noted. Also, relevance of these ideas to statistical estimation of time series is discussed briefly.

1. Quadrature Formulas and Monosplines

Schoenberg [11] pointed out an enticing correspondence between quadrature formulas and monosplines. A *monospline of degree m* with knots $\{\xi_k\}_1^r$ is an expression of the form

$$M(x) = \frac{x^m}{m!} + \sum_{\nu=0}^{m-1} b_\nu x^\nu + \sum_{\nu=1}^{r} d_\nu (x - \xi_\nu)_+^{m-1}, \tag{1.1}$$

where b_ν and d_k are real. We designate this class of monosplines by $\mathscr{M}_{m,r}(\{\xi_\nu\}_1^r)$ and, where no ambiguity will arise, the display of the given knots $\{\xi_\nu\}_1^r$ will be suppressed. When the term $x^m/m!$ is discarded in (1.1) we refer to the resultant function as *a spline of degree* $m - 1$ with knots $\{\xi_\nu\}_1^r$. The linear space of these functions is denoted by $\mathscr{S}_{m,r}(\{\xi_\nu\}_1^r) = \mathscr{S}_{m,r}$.

We review Schoenberg's reasoning. Let f be of continuity class $C^{(n)}$, on [0,1]. Integrating by parts, $n - 1$ times yields

$$\int_0^1 M(x) f^{(n)}(x)\, dx$$
$$= \sum_{j=0}^{n-2} (-1)^j f^{(n-1-j)}(x)\, M^{(j)}(x)\, \Big|_0^1 + (-1)^{n-1} \int_0^1 M^{(n-1)}(x) f'(x)\, dx. \tag{1.2}$$

Set $\xi_0 = 0$, $\xi_{r+1} = 1$. Then

$$M^{(n-1)}(x) = x + (n-1)!\, b_{n-1} + \sum_{k=1}^{p} (n-1)!\, d_k$$
$$\equiv x + \alpha_p, \qquad \xi_p < x < \xi_{p+1}, \qquad p = 0, 1, \ldots, r.$$

Substitution for $M^{(n-1)}(x)$ in (1.2) and another integration by parts of $\int_0^1 x f'(x)\, dx$ finally produce the formula

$$\int_0^1 f(x)\, dx = \sum_{j=0}^{n-1} B_j f^{(j)}(1) + \sum_{j=0}^{n-1} A_j f^{(j)}(0)$$
$$+ \sum_{\nu=1}^{r} c_\nu f(\xi_\nu) + (-1)^n \int_0^1 M(x) f^{(n)}(x)\, dx, \tag{1.3}$$

where

$$B_j = (-1)^j M^{(n-j-1)}(1), \qquad j = 0, 1, \ldots, n-1, \tag{1.4}$$
$$A_j = (-1)^{j+1} M^{(n-j-1)}(0),$$

and

$$c_\nu = M^{(n-1)}(\xi_\nu-) - M^{(n-1)}(\xi_\nu+), \qquad \nu = 1, 2, \ldots, r.$$

Set

$$Q(f) = \sum_{j=0}^{n-1} B_j f^{(j)}(1) + \sum_{j=0}^{n-1} A_j f^{(j)}(0) + \sum_{\nu=1}^{r} c_\nu f(\xi_\nu) \tag{1.5}$$

and

$$R(f) = (-1)^n \int_0^1 M(x) f^{(n)}(x)\, dx. \tag{1.5'}$$

Inspection of (1.3) reveals the equality $Q(f) = \int_0^1 f(x)\, dx$ holding whenver f is a polynomial of degree at most $n - 1$.

Note that, if $M(x)$ is replaced by a spline $S(x)$ of degree $n - 1$ with knots $\{\xi_\nu\}_1^r$, then the term $\int_0^1 f(x)\, dx$ does not arise and (1.3) reduces to

$$Q(f) + (-1)^n \int_0^1 S(x) f^{(n)}(x)\, dx = 0. \tag{1.6}$$

Of course, the evaluations of (1.4) are then construed with $S(x)$ inserted for $M(x)$.

The foregoing analysis demonstrated that every monospline of degree n induces a quadrature formula of the form (1.3), satisfying $R(f) = 0$ for f a polynomial of degree at most $n - 1$.

The converse is valid. Specifically, let $Q(f)$ be a quadrature formula of the type (1.5) *with the property that*

$$L(f) = \int_0^1 f(x)\, dx = Q(f) \tag{1.7}$$

for f a polynomial of degree $\leqslant n - 1$. We will establish the existence of a monospline $M(x)$ of degree n with knots $\{\xi_\nu\}_1^r$ such that $R(f) = L(f) - Q(f)$ admits the representation given in (1.5').

Let f be of continuity class $C^{(n)}[0, 1]$. Taylor's formula with remainder applies to give

$$f(x) = \sum_{i=0}^{n-1} f^{(i)}(0) \frac{x^i}{i!} + \frac{1}{(n-1)!} \int_0^1 f^{(n)}(t)(x - t)_+^{n-1}\, dt.$$

Since $R(f) = L(f) - Q(f)$ is a continuous linear functional annihilating polynomials of degree $\leqslant n-1$, we obtain

$$R(f) = \frac{1}{(n-1)!} \int_0^1 f^{(n)}(t)\, R_x(x-t)_+^{n-1}\, dt.$$

Observe for $R(f) = \int_0^1 f(x)\, dx - Q(f)$ that

$$\begin{aligned} \frac{(-1)^n}{(n-1)!} R_x(x-t)_+^{n-1} &= \frac{(-1)^n}{(n-1)!} \left[\int_0^1 (x-t)_+^{n-1}\, dx - Q_x(x-t)_+^{n-1}\right] \\ &= \frac{(-1)^n}{n!} (1-t)^n - (-1)^n \sum_{j=0}^{n-1} B_j \frac{(1-t)^{n-1-j}}{(n-1-j)!} \\ &\quad - (-1)^n \sum_{\nu=1}^{r} \frac{c_\nu}{(n-1)!} (\xi_\nu - t)_+^{n-1} \end{aligned} \tag{1.8}$$

and the right side is manifestly a monospline $M(t)$ of degree n of the desired kind. The above discussion can be summarized as follows:

THEOREM 1.1. *Every quadrature formula of the form* (1.5) *for the functional* $L(f) = \int_0^1 f(x)\, dx$ *which satisfies* (1.7) *has a remainder functional*

$$R(f) = (-1)^n \int_0^1 f^{(n)}(x)\, M(x)\, dx \tag{1.9}$$

for some monospline $M(x)$ of degree n. Conversely, every monospline of degree n induces a quadrature formula of the type (1.5) *and the identity* (1.3) *prevails. The coefficients of the quadrature formula are computed according to the relations* (1.4). *The correspondence between monosplines and quadrature formulas is* 1 : 1 *as is evident from the representation* (1.9).

As we stated earlier, Theorem 1.1 is due to Schoenberg [13].

Specialized quadrature formulas of the types (0.5) and (0.4) emanate by requiring that the monospline obey certain boundary conditions. Some examples are considered in Sections 2 and 3.

Now, the objective of our endeavors is to characterize quadrature formulas (or, equivalently, corresponding monosplines) best in the sense of Sard (cf. (0.1) and (0.2)). To this end, observe first that application of the Schwartz inequality (together with the condition for equality) in (1.9) furnishes the relation

$$\sup_{\|f\|_n=1} |\, R(f)|^2 = \int_0^1 |\, M(x)|^2\, dx, \tag{1.10}$$

where $\|f\|_n$ is as defined following (0.2). Therefore, the determination of the best quadrature formula obviously reduces to evaluating

$$\min_{M\in\mathcal{M}_{n,r}} \int_0^1 | M(x)|^2 \, dx = \int_0^1 | M_*(x)|^2 \, dx \tag{1.11}$$

and characterizing $M_*(x)$, where the minimum is extended over all monosplines satisfying the appropriate boundary conditions with the given knots. Theorem 1.1 assures that this class is nonvoid provided only that a quadrature formula is available for which (1.7) is satisfied. This requirement entails in some cases assumptions concerning prescription of the number of knots and the nature of the boundary conditions.

To solve the minimum problem (1.11) is obviously the same as finding the best approximation to the function $x^n/n!$ in $L^2(0, 1)$ by splines in $\mathcal{S}_{n,r}$ restricted by certain linear constraints induced by the boundary conditions. The corresponding set of monosplines of degree n with fixed knots $\{\xi_\nu\}_1^r$ comprises a convex set in the Hilbert space $L^2(0, 1)$. Since $L^2(0, 1)$ is strictly convex the minimizing monospline M is uniquely determined.

Let $S(x)$ be any spline of degree $n - 1$ with knots $\{\xi_k\}_1^r$ satisfying the relevant boundary conditions. If $M(x)$ is a monospline of degree n with the required properties, then $M(x) + S(x)$ is a monospline of the same class. The standard Hilbert space variational argument shows that $M_*(x)$ solves the minimization problem if and only if

$$\int_0^1 M_*(x)\, S(x)\, dx = 0 \tag{1.12}$$

for all splines $S(x)$ of degree $n - 1$ with knots $\{\xi_k\}_1^r$ satisfying the appropriate boundary conditions. Therefore, to secure the minimizing M_* in (1.11), it suffices to exhibit a monospline of degree n satisfying the relevant boundary conditions, and (1.12).

The previous analysis provides the proof of the following elementary variational theorem.

THEOREM 1.2. *Suppose there exists a quadrature formula of the desired kind satisfying* (1.7). *The monospline $M_*(x)$ corresponding to the best quadrature formula is uniquely determined by the prescription that $M_*(x)$ fulfills certain boundary conditions and satisfies the orthogonality relations* (1.12) *with respect to all splines displaying the same knots and satisfying the same boundary conditions.*

The description of the best $M_*(x)$ given in Theorem 1.2 is not of practical utility.

More practically, the best $M_*(x)$ can be obtained by solving an appropriate linear system of equations. In all the examples of Section 2–5, $M_*(x)$ is effectively evaluated from the solution of a simple interpolation problem. Specifically, $M_*(x)$ is determined to satisfy the boundary conditions (those generating the desired kind of quadrature formulas), additional "adjoint" boundary constraints (see Section 4), and "interpolation" conditions. The total number of conditions equals the number of parameters for the monospline.

THEOREM 1.3. *The collection of admissible quadrature formulas $\mathscr{C}$ is nonvoid if and only if the determinant of the linear equations characterizing the best monospline is nonzero.*

Proof. When $\mathscr{C}$ is nonvoid, it was pointed out, the best quadrature formula can also be characterized in terms of a best approximation problem in $L_2[0, 1]$. Since L_2 is strictly convex, the solution M_* is unique. As indicated just prior to this theorem the best monospline can be discerned from the solution of a system of linear equations. The associated homogeneous system of equations yields all spline functions fulfilling the same homogeneous boundary and "interpolation" conditions. If this system possesses a nontrivial solution (i.e., if the determinant of the linear system is zero), we obtain a nontrivial spline $\tilde{S}$ satisfying the same boundary conditions and the orthogonality property (1.12). It follows that $\tilde{S}$ is orthogonal to itself and therefore $\tilde{S} = 0$, a centradiction. Conversely, if the determinant is nonzero, then we can construct an admissible monospline by solving an appropriate interpolation problem, and $\mathscr{C}$ is nonvoid. This completes the proof of Theorem 1.3.

In the following sections, several important concrete examples of the best quadrature formulas and their complete characterizations will be elaborated.

2. SOME IMPORTANT CLASSES OF QUADRATURE FORMULAS

For the explicit characterization of the monospline generating the best quadrature formula in the sense of Sard, with $L(f) = \int_0^1 f(x)\,dx$, it is useful to write every admissible monospline of degree n as the nth derivative of a monospline of degree $2n$, $N^{(n)}(x) = M(x)$ where now

$$N(x) = \frac{x^{2n}}{2n!} + \sum_{i=0}^{2n-1} e_i x^i + \sum_{j=1}^{r} a_j (x - \xi_j)_+^{2n-1}. \tag{2.1}$$

This amounts merely to a change of notation in (1.9) and will prove most convenient in discerning the properties of the "best" monospline associated with a given type quadrature formula. (Notice that $N(x)$ has $2n + r$ parameters which is the size of the linear system alluded to prior to Theorem 1.3).

In the new notation, the relation (1.3) reads as

$$L(f) = \int_0^1 f(x)\,dx = Q(f) + R(f) \tag{2.2}$$

for all f of continuity class $C^n[0, 1]$, where $Q(f)$ has the representation (1.5) as earlier but the coefficients are expressed as

$$\begin{aligned} A_j &= (-1)^{j+1} N^{(2n-j-1)}(0), \\ B_j &= (-1)^j N^{(2n-j-1)}(1), \qquad j = 0, 1, 2,..., n-1, \end{aligned} \tag{2.3}$$

$$c_\mu = N^{(2n-1)}(\xi_\mu -) - N^{(2n-1)}(\xi_\mu +), \qquad \mu = 1, 2,..., r, \tag{2.4}$$

and the remainder functional takes the form

$$R(f) = (-1)^n \int_0^1 N^{(n)}(x) f^{(n)}(x)\,dx. \tag{2.5}$$

The variational argument described in connection with Theorem 1.2 shows that the quadrature formula best in the sense of Sard is induced by that monospline $N_*(x)$ of degree $2n$ with knots $\{\xi_\nu\}_1^r$ satisfying

$$\int_0^1 N_*^{(n)}(x)\, S^{(n)}(x)\,dx = 0 \tag{2.6}$$

for all splines $S(x)$ of degree $2n - 1$ with knots $\{\xi_\nu\}_1^r$; both $N_*(x)$ and $S(x)$ will be required to satisfy certain boundary conditions in order that the quadrature formula be of the requisite form.

EXAMPLE 1. Consider quadrature formulas of the type

$$Q(f) = \sum_{\nu=1}^r c_\nu f(\xi_\nu). \tag{2.7}$$

Comparing with (1.3) and taking cognizance of (2.3), we infer that the corresponding class of monosplines of degree $2n$ must satisfy

$$\begin{aligned} N^{(n)}(0) &= N^{(n+1)}(0) = \cdots = N^{(2n-1)}(0) = 0, \\ N^{(n)}(1) &= N^{(n+1)}(1) = \cdots = N^{(2n-1)}(1) = 0. \end{aligned} \tag{2.8}$$

A simple analysis reveals that quadrature formulas of the kind (2.7) exist satisfying the requirement $R(f) = 0$ for f a polynomial of degree $\leqslant n - 1$ if

$$r \geqslant n. \tag{2.9}$$

Assume condition (2.9) *holds.* Let $S(x)$ be an arbitrary spline of degree $2n - 1$ with knots $\{\xi_v\}_1^r$ which also satisfies the boundary conditions (2.8).

The collection

$$\tilde{\mathscr{C}} = \{N(x) = \tilde{N}(x) + S(x)\}, \tag{2.10}$$

where $\tilde{N}(x)$ is a fixed monospline and $S(x)$ an arbitrary spline of the proper degree fulfilling the boundary constraints (2.8), spans the set of all admissible monosplines inducing quadrature formulas of type (2.7). The set of all $S(x)$ as described above is denoted by

$$\mathscr{C} = \{S(x)\}.$$

The variational principle affirms that $N_*(x)$ minimizing $\int_0^1 | N^{(n)}(x)|^2 \, dx$ for N traversing $\tilde{\mathscr{C}}$ is uniquely determined by the properties

$$N_* \text{ satisfies (2.8)} \tag{2.11}$$

and the orthogonality condition

$$\int_0^1 N_*^{(n)}(x) \, S^{(n)}(x) \, dx = 0 \qquad \text{for all } S \in \mathscr{C}. \tag{2.12}$$

We claim that $N_*(x)$ satisfies

$$\begin{aligned} N_*^{(n)}(0) &= N_*^{(n+1)}(0) = \cdots = N_*^{(2n-1)}(0) = 0, \\ N_*^{(n)}(1) &= N_*^{(n+1)}(1) = \cdots = N_*^{(2n-1)}(1) = 0, \end{aligned} \tag{2.13}$$

and the further interpolation conditions

$$N_*(\xi_1) = N_*(\xi_2) = \cdots = N_*(\xi_r) = 0. \tag{2.14}$$

The interpolation Theorem 2 of Karlin [4] tells us that $N_*(x)$ is uniquely determined by these stipulations. A formal proof that $N_*(x)$ satisfying(2.13) and (2.14) implies the validity of (2.12) runs as follows:

Applying the identity (1.3) with $f(x) = N_*(x)$, referring to (1.6), and with $S(x)$ playing the role of $N(x)$, we infer the relation

$$(-1)^n \int_0^1 S^{(n)}(x)\, N_*^{(n)}(x)\, dx = \sum_{j=0}^{n-1} (-1)^{j+1} S^{(2n-j-1)}(1)\, N_*^{(j)}(1) + \sum_{j=0}^{n-1} (-1)^j S^{(2n-j-1)}(0)\, N_*^{(j)}(0) + \sum_{\nu=1}^{r} a_\nu N_*(\xi_\nu). \tag{2.14a}$$

Since $S(x)$ satisfies the boundary constraints (2.13), and $N_*(x)$ the interpolatory conditions (2.14), we find that all terms on the right in (2.14a) vanish. Thus, (2.12) prevails for all $S \in \mathscr{C}$, as is desired to be shown.

EXAMPLE 2. Consider quadrature formulas of the type

$$Q(f) = \sum_{k=1}^{r} c_k f(\xi_k) + \sum_{\mu=1}^{n-p} A_\mu f^{(i_\mu')}(0) + \sum_{\nu=1}^{n-q} B_\nu f^{(j_\nu')}(1), \qquad p, q \leqslant n, \tag{2.15}$$

where $\{i_\mu'\}_1^{n-p}$ and $\{j_\nu'\}_1^{n-q}$ are prescribed such that

$$0 \leqslant i_1' < i_2' < \cdots < i_{n-p}' \leqslant n-1; \quad 0 \leqslant j_1' < j_2' < \cdots < j_{n-q}' \leqslant n-1$$

(if $p = n$ ($q = n$), then the second (third) sum is missing). We wish to determine that quadrature formula of the form (2.15) which is the best in the sense of Sard. Consulting (1.3) in conjunction with Theorem 1.1, we infer that the corresponding monosplines (of degree $2n$) must satisfy the boundary conditions

$$\begin{aligned} N^{(k_1)}(0) &= N^{(k_2)}(0) = \cdots = N^{(k_p)}(0) = 0, \\ N^{(l_1)}(1) &= N^{(l_2)}(1) = \cdots = N^{(l_q)}(1) = 0, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} k_\mu &= 2n - 1 - i_\mu, \qquad \mu = 1, 2,\ldots, p, \\ l_\nu &= 2n - 1 - j_\nu, \qquad \nu = 1, 2,\ldots, q, \end{aligned} \tag{2.17}$$

and $\{j_\mu\}_1^q$ and $\{i_\nu\}_1^p$ are the sets complementary to $\{j_\mu'\}_1^{n-q}$ and $\{i_\nu'\}_1^{n-p}$, respectively, in $\{0, 1,\ldots, n-1\}$. Note that

$$\begin{aligned} n \leqslant k_1 < k_2 < \cdots < k_p \leqslant 2n - 1, \\ n \leqslant l_1 < l_2 < \cdots < l_q \leqslant 2n - 1. \end{aligned}$$

Examination of the integration by parts formula (consult also (2.14a)) suggests that the optimal $N_*(x)$ should satisfy the "adjoint boundary conditions"

$$\begin{aligned} N_*^{(i_\nu')}(0) &= 0, \qquad \nu = 1, 2,..., n - p, \\ N_*^{(j_\mu')}(1) &= 0, \qquad \mu = 1, 2,..., n - q, \end{aligned} \tag{2.18}$$

and the interpolatory conditions

$$N_*(\xi_p) = 0, \qquad p = 1, 2,..., r. \tag{2.19}$$

The existence of a monospline obeying the constraints (2.16) and fulfilling the additional conditions (2.18) and (2.19) is not guaranteed. A necessary and sufficient conditions for the existence of a unique monospline satisfying (2.16)–(2.19) is (the Fredholm determinant involved is that based on the kernel $K(z, w)$ of Karlin [4]; notation is that of [3]; see also (2.17)):

$$K\begin{pmatrix} \xi_1, ..., \xi_r, j_1', ..., j_{n-q}', l_1, ..., l_q \\ i_1, ..., i_p, k_1', ..., k_{n-p}', \xi_1, ..., \xi_r \end{pmatrix} > 0, \tag{2.20}$$

where $\{k_\nu'\}_1^{n-p}$ is the set complimentary to $\{k_\nu\}_1^p$ from $\{n, n+1,..., 2n-1\}$. The precise requirements on the indices, j''s, l's, i's and k''s for which (2.20) is positive are recorded in Theorem 1 of Karlin [4]. In particular, if $r \geqslant n$, then (2.20) always prevails. When $p \leqslant r < n$, then (2.20) holds if and only if $k_{\nu+r-p}' \geqslant j_\nu'$, $\nu = 1, 2,..., n - r$. Analogous requirements apply in the other cases.

It is worth emphasizing that the existence of a monospline satisfying (2.16)–(2.19) is equivalent to the existence of a quadrature formula of the type (2.15) for which (0.1) holds (see Theorem 1.3).

The verification that N_* fulfilling (2.16)–(2.19) also inherits the orthogonality property

$$\int_0^1 N_*^{(n)}(x)\, S^{(n)}(x)\, dx = 0$$

with respect to all splines $S(x)$ satisfying the constraints (2.16) is accomplished as in Example 1 mutatis mutandis.

3. Best Quadrature Formulas with General Boundary Conditions

Consider quadrature formulas of the form

$$Q(f) = \sum_{i=1}^{p} a_i U_i(f) + \sum_{i=1}^{q} b_i V_i(f) + \sum_{k=1}^{r} c_k f(\xi_k), \tag{3.1}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_r < 1$,

$$U_i(f) = \sum_{j=1}^{n} A_{ij} f^{(j-1)}(0), \qquad i = 1, 2, \ldots, p, \tag{3.2}$$

$$V_i(f) = \sum_{j=1}^{n} B_{ij} f^{(j-1)}(1), \qquad i = 1, 2, \ldots, q. \tag{3.3}$$

We assume that the A_{ij} and B_{ij} are real, that $0 \leqslant p, q \leqslant n$, and that

$$\operatorname{rank} \| A_{ij} \| = p; \qquad \operatorname{rank} \| B_{ij} \| = q. \tag{3.4}$$

By adding further relations to those of (3.2) and (3.3) totalling n for each boundary point, we can, in a standard way, construct adjoint boundary linear forms (see Neumark [6, Chap. 1]). Specifically, first expand the matrix A to an $n \times n$ nonsingular matrix $\hat{A}$. This is feasible by virtue of the rank stipulations (3.4). Let $\tilde{A} = \hat{A}^{-1} = \| \tilde{a}_{ij} \|_1^n$. Then the vector relation

$$\bar{f}(0) = \tilde{A}\bar{u}(f) \tag{3.5}$$

obtains, where $\bar{u}(f)$ and $\bar{f}(0)$ denote, respectively, the vectors

$$\bar{u}(f) = (U_1(f), U_2(f), \ldots, U_n(f)), \qquad \bar{f}(0) = (f(0), f^{(1)}(0), f^{(2)}(0), \ldots, f^{(n-1)}(0)).$$

Substituting from (3.5) into the boundary terms of the integration by parts formula (1.3), we get

$$\begin{aligned} \sum_{i=1}^{n} (-1)^i M^{(n-i)}(0) f^{(i-1)}(0) &= \sum_{i=1}^{n} (-1)^i M^{(n-i)}(0) \sum_{j=1}^{n} \tilde{a}_{ij} U_j(f) \\ &= \sum_{j=1}^{n} U_j(f)\, \tilde{U}_j(M) = B(f, M), \end{aligned} \tag{3.6}$$

where

$$\tilde{U}_j(M) = \sum_{i=1}^{n} (-1)^{n+1-i} \tilde{a}_{n+1-i,j} M^{(i-1)}(0), \qquad j = 1, 2, \ldots, n.$$

Adjoint boundary conditions to

$$U_i(f) = 0, \qquad i = 1, 2, \ldots, p, \tag{3.7}$$

at the endpoint 0 are taken to be

$$\tilde{U}_j(f) = 0, \qquad j = p + 1, p + 2, \ldots, n. \tag{3.8}$$

The justification of the definition in (3.8) rests on the property that $B(f, M)$ is a bilinear form in the vector variables $\bar{f}$ and $\overline{M}$ which vanishes when f satisfies the boundary conditions (3.7) and M satisfies the adjoint boundary conditions (3.8).

The $n - p \times n$ matrix associated with the adjoint boundary forms is

$$\|(-1)^{n+1-i}\, \tilde{a}_{n+1-i,j}\, \|_{j=p+1,\,i=1}^{n \quad\ n} . \tag{3.9}$$

Since $\tilde{A} = \hat{A}^{-1}$ is nonsingular, it is easily verified that the matrix (3.9) has full rank (i.e., rank $n - p$).

A similar procedure can be implemented with regard to the boundary point $x = 1$. Thus, extend the matrix B to an $n \times n$ nonsingular matrix $\hat{B}$ and let $\tilde{B} = \hat{B}^{-1}$. A set of adjoint boundary forms at the endpoint 1 are then cast explicitly as

$$\tilde{V}_j(M) = \sum_{i=1}^{n} (-1)^{n-i}\, \tilde{b}_{n+1-i,j} M^{(i-1)}(1), \qquad j = 1, 2,\dots, n, \tag{3.10}$$

so that

$$\sum_{j=1}^{n} (-1)^{j+1}\, M^{(n-j)}(1)\, f^{(j-1)}(1) = \sum_{i=1}^{n} V_i(f)\, \tilde{V}_i(M) \tag{3.11}$$

and the adjoint boundary conditions to

$$V_i(f) = 0, \qquad i = 1, 2,\dots, q, \tag{3.12}$$

at $x = 1$ are then defined to be

$$\tilde{V}_i(f) = 0, \qquad i = q + 1, q + 2,\dots, n. \tag{3.13}$$

It follows as before that

$$\operatorname{rank}(\|(-1)^{n+i}\, \tilde{b}_{n+1-i,j}\, \|_{j=q+1,\,i=1}^{n \quad\ n}) = n - q.$$

Remark. The above construction of adjoint boundary forms is independent of the determination of the specific expanded matrices $\hat{A}$ and $\hat{B}$ in the sense that any two sets of adjoint boundary conditions are connected by a nonsingular linear transformation and, in particular, total the same number of conditions and define the same linear subspaces (see Neumark [6]).

Let $M(x)$ be a monospline of degree n with knots $\{\xi_u\}_1^r$. In view of (3.6) and (3.10), the integration by parts identity (1.3) can be written equivalently as

$$\int_0^1 f(x)\,dx = \sum_{i=1}^{n} U_i(f)\,\tilde{U}_i(M) + \sum_{i=1}^{n} V_i(f)\,\tilde{V}_i(M) + \sum_{k=1}^{r} c_k f(\xi_k) + (-1)^n \int_0^1 M(x) f^{(n)}(x)\,dx \tag{3.14}$$

valid for f of continuity class $C^{(n)}[0, 1]$.

Comparing (3.1) with (3.14) and reasoning as in Theorem 1.1, we establish a 1 : 1 correspondence between the collection of quadrature formulas of type (3.1) satisfying $\int_0^1 f(x)\,dx = Q(f)$ for all polynomials of degree at most $n-1$ and the set of all monosplines $M(x)$ of degree n with knots $\{\xi_u\}_1^r$ fulfilling the boundary constraints

$$\begin{aligned} \tilde{U}_i(M) &= 0, \qquad i = p+1, p+2,..., n, \\ \tilde{V}_i(M) &= 0, \qquad i = q+1, q+2,..., n. \end{aligned} \tag{3.15}$$

We designate this class of monosplines as $\mathscr{M}_{n,r}(\{\xi_u\}; \mathscr{B})$.

Sufficient conditions to assure that the collection $\mathscr{M}_{n,r}(\{\xi_u\}; \mathscr{B})$ is nonvoid are indicated later in Theorem 3.2.

In characterizing the quadrature formula of type (3.1) best in the sense of Sard it is convenient to change notation by expressing the general monospline $M(x)$ in $\mathscr{M}_{n,r}(\{\xi_u\}; \mathscr{B})$ as the nth derivative of a monospline $N(x)$ of degree $2n$, $M(x) = N^{(n)}(x)$ (as previously, cf. Section 2). The formulas are altered as follows: (3.14) becomes

$$\int_0^1 f(x)\,dx = \sum_{i=1}^{n} U_i(f)\,\tilde{U}_i(N) + \sum_{i=1}^{n} V_i(f)\,\tilde{V}_i(N) + \sum_{k=1}^{r} c_k f(\xi_k) + (-1)^n \int_0^1 N^{(n)}(x) f^{(n)}(x)\,dx, \tag{3.16}$$

where, with $\hat{A}^{-1} = \tilde{A} = \| \tilde{a}_{i,j} \|_1^n$, $\hat{B}^{-1} = \tilde{B} = \| \tilde{b}_{ij} \|_1^n$,

$$\begin{aligned} \tilde{U}_j(N) &= \sum_{i=1}^{n} (-1)^{n+i-1}\, \tilde{a}_{n+1-i,j} N^{(n+i-1)}(0), \qquad j = 1, 2,..., n, \\ \tilde{V}_j(N) &= \sum_{i=1}^{n} (-1)^{n+i}\, \tilde{b}_{n+1-i,j} N^{(n+i-1)}(1), \qquad j = 1, 2,..., n, \end{aligned} \tag{3.17}$$

and the boundary forms U_i and V_i remain as defined in (3.2) and (3.3) (the c_k are defined in 2.4).

Substituting $N(x)$, a monospline of degree $2n$, for $f(x)$, and $S(x)$, a spline of degree $2n-1$ with knots $\{\xi_k\}_1^r$, (3.16) reduces to (compare to (1.6))

$$\int_0^1 N^{(n)}(x)\, S^{(n)}(x)\, dx = (-1)^{n+1} \sum_{j=1}^{n} U_j(N)\, \tilde{U}_j(S) + (-1)^{n+1} \sum_{j=1}^{n} V_j(N)\, \tilde{V}_j(S) + (-1)^{n+1} \sum_{k=1}^{r} c_k N(\xi_k). \tag{3.18}$$

Inspection of (3.18) implies the following theorem:

THEOREM 3.1. *Suppose there exists a monospline $N(x)$ of degree $2n$ with knots $\{\xi_k\}_1^r$ satisfying the adjoint boundary constraints*

$$\begin{aligned} \tilde{U}_i(N) &= 0, \qquad i = p+1,\ldots, n,\\ \tilde{V}_i(N) &= 0, \qquad i = q+1,\ldots, n. \end{aligned} \tag{3.19}$$

(Recall that this class of monosplines is designated as $\mathcal{M}_{2n,r}(\{\xi_u\}; \mathcal{B})$.) Suppose there exists a monospline $N_*(x)$ in $\mathcal{M}_{2n,r}(\{\xi_u\}; \mathcal{B})$ satisfying the additional boundary conditions

$$\begin{aligned} U_i(N_*) &= 0, \qquad i = 1, 2,\ldots, p,\\ V_i(N_*) &= 0, \qquad i = 1, 2,\ldots, q, \end{aligned} \tag{3.20}$$

and the interpolation requirements

$$N_*(\xi_k) = 0, \qquad k = 1, 2,\ldots, r, \tag{3.21}$$

then this monospline is best, in the sense of Sard, for the linear functional $L(f) = \int_0^1 f(x)\, dx$ among all quadrature formulas of the type (3.1) *induced by monosplines of class $\mathcal{M}_{2n,r}(\{\xi_u\}; \mathcal{B})$.*

The conditions (3.19)–(3.21) are equivalent to the orthogonality property

$$\int_0^1 N_*^{(n)}(x)\, S^{(n)}(x)\, dx = 0$$

valid for all splines $S(x)$ of degree $2n-1$ with knots $\{\xi_k\}_1^r$ satisfying the adjoint boundary conditions (3.19).

Appealing to the general interpolation Theorem 2 of Karlin [4], we deduce in Theorem 3.2 below a sufficient condition for the existence of a monospline satisfying (3.19)–(3.21). This monospline is best in the sense of Sard.

THEOREM 3.2. *Let* $C = \| c_{ij} \|_{i=1,j=1}^{n,\ 2n}$ *be the matrix of the boundary forms* $\{U_i\}_1^p$ *and* $\{\tilde{U}_i\}_{p+1}^n$, *and* $D = \| d_{ij} \|_{i=1,j=1}^{n,\ 2n}$ *the matrix of the boundary forms* $\{V_i\}_1^q$ *and* $\{\tilde{V}_i\}_{q+1}^n$. *Suppose* $\tilde{C} = \|(-1)^j c_{ij}\|$ *is sign consistent of order* n (cf. [3]), i.e., *all non zero* n-th *order subdeterminants of* $\tilde{C}$ *have a fixed sign and rank* $(\tilde{C}) = n$. *Assume also that* D *is sign consistent of order* n *and rank* $D = n$. *Then there exists a unique monospline* $N_*(x)$ *of degree* $2n$ *with knots* $\{\xi_k\}_1^r$ *satisfying* (3.19)–(3.21).

Remark. Checking the sign consistency requirements on the matrices $\tilde{C}$ and D (defined in Theorem 3.2) is facilitated with the help of the following facts. Note that the matrix C of the boundary forms $\{U_i\}$ and $\{\tilde{U}_i\}$ has the structure

$$C = \begin{Vmatrix} \| A_{ij} \|_{i=1,j=1}^{p,\ n} & 0_{p\times n} \\ 0_{n-p\times n} & \|(-1)^{n+1-j}\, \tilde{a}_{n+1-j,i} \|_{i=p+1,j=1}^{n,\ n} \end{Vmatrix}. \tag{3.22}$$

By hypothesis, rank $(\| A_{ij} \|) = p$. We pointed out earlier, following (3.9), that the lower right matrix has rank $n - p$. Therefore, C has rank n. From the definition, we have

$$\tilde{C} = \begin{Vmatrix} \| A_{ij}(-1)^j \|_{i=1,j=1}^{p,\ n} & 0_{p\times n} \\ 0_{n-p\times n} & \|(-1)\, \tilde{a}_{n+1-j,i} \|_{i=p+1,j=1}^{n,\ n} \end{Vmatrix}. \tag{3.23}$$

It is easy to see that the only nonzero subdeterminants of $\tilde{C}$ of order n are those with their first p columns selected from the first n columns of (3.23) and their last $n - p$ columns chosen from the last n columns. It follows that

$$\tilde{C} \text{ is sign consistent of order } n\ (SC_n) \tag{3.24}$$

if and only if

$$\|(-1)^j A_{ij} \|_{i=1,j=1}^{p,\ n} \quad \text{is} \quad SC_p, \tag{3.25a}$$

$$\| \tilde{a}_{n+1-j,i} \|_{i=p+1,j=1}^{n,\ n} \quad \text{is} \quad SC_{n-p}. \tag{3.25b}$$

In a similar manner we find that the matrix D associated with the boundary forms $\{V_i\}$ and $\{\tilde{V}_i\}$ at the endpoint 1 is SC_n iff

$$\| B_{ij} \|_{i=1,j=1}^{q,\ n} \quad \text{is} \quad SC_q, \tag{3.26a}$$

$$\|(-1)^j\, \tilde{b}_{n+1-j,i} \|_{i=q+1,j=1}^{n,\ n} \quad \text{is} \quad SC_{n-q} \tag{3.26b}$$

(recall that $\| B_{ij} \|$ is the matrix of the boundary forms $\{V_i\}_1^q$ and $\hat{B}^{-1} = \| \tilde{b}_{ij} \|_1^n$).

In the next section a class of important examples of certain boundary forms fulfilling the requirements of Theorems 3.1 and 3.2 is set forth.

4. Special Boundary Forms for Vibrating Systems

An important example for Theorems 3.1 and 3.2 arises from the boundary forms

$$U_i(f) = f^{(i-1)}(0) + (-1)^{n+p+i-1}\, c_i f^{(n-i)}(0), \qquad i = 1, 2,\dots, p, \quad (4.1a)$$

$$V_i(f) = f^{(i-1)}(1) + (-1)^{q+i}\, d_i f^{(n-i)}(1), \qquad i = 1, 2,\dots, q, \quad (4.1b)$$

where $0 \leqslant p,\, q \leqslant n$, *$n$ is an even integer* and the c_i and d_i are strictly positive. Boundary conditions of the kind (4.1) occur in physical problems associated with vibrating segments; see Karlin [3, Chap. 10, §7].

The $p \times n$ matrix for the boundary forms $\{U_i\}_1^p$ has the representation (when $p \leqslant n - p$)

$$A = \left\| \begin{array}{cccccccccc} 1 & 0 & & \cdots & & 0 & & c_1(-1)^{n+p} & & \\ 0 & 1 & 0 & \cdots & 0 & & c_2(-1)^{n+p+1} & & 0 & \\ \vdots & \ddots & \ddots & \ddots & & \iddots & \iddots & & \iddots & \vdots \\ 0 \cdots 0 & & 1 & 0 \cdots 0 & & c_p(-1)^{n+2p-1} & & 0 & & 0 \end{array} \right\| \qquad (4.2a)$$

with 1's running down the main diagonal and

$$(-1)^{n+p}c_1\,, (-1)^{n+p+1}c_2\,,\dots, (-1)^{n+2p-1}c_p$$

occurring on the skew diagonal. When $p > n - p$ the skew and main diagonals intertwine with no ambiguity present since n is even,[2] and then the boundary forms (4.1a) generate the matrix

$$A = \left\| \begin{array}{ccccccccc} 1 & 0 & & \cdots & & & 0 & c_1(-1)^{n+p} & \\ 0 & 1 & 0 & \cdots & 0 & & c_2(-1)^{n+p+1} & & 0 \\ \vdots & & \ddots & & & \iddots & & & \vdots \\ & & & \ddots & \iddots & & & & \\ & & & \iddots & \ddots & & & & \\ 0 \cdots 0 & & c_p(-1)^{n+2p-1} & 0 \cdots 0 & & 1 & 0 & \cdots & 0 \end{array} \right\| \qquad (4.2b)$$

It is a simple matter to check that A has rank p.

[2] The subsequent analysis could be carried out with n odd involving unessential technical modifications.

In line with the procedure outlined in Section 3, we expand the matrix A to an $n \times n$ matrix $\hat{A}$ of full rank. Indeed, for the case at hand, $\hat{A}$ is chosen to be of the same form as A. Specifically, we take

$$\hat{A} = \left\| \begin{matrix} 1 & 0 & 0 \cdots 0 & 0 & c_1(-1)^{n+p} \\ 0 & 1 & 0 \cdots 0 & c_2(-1)^{n+p+1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & (-1)^{2n+p-2} & 0 \cdots 0 & 1 & 0 \\ (-1)^{2n+p-1} & 0 & 0 \cdots 0 & 0 & 1 \end{matrix} \right\| \tag{4.3}$$

where $\hat{A}$ has 1's on the main diagonal, the entries

$$(-1)^{n+p}c_1,\ (-1)^{n+p+1}c_2, \ldots, (-1)^{n+2p-1}c_p,\ (-1)^{n+2p}, \ldots, (-1)^{2n+p-1}$$

running down the skew diagonal and 0 terms elsewhere.

It will become evident that the c_i play no role in the reasoning so long as they are strictly positive. Henceforth, whenever convenient we set $c_i = 1$ for all i, and similarly for d_i.

The properties of the matrices $\hat{A}$ and $\tilde{A} = \hat{A}^{-1}$ to be established are listed below.

Proposition 4.1. *The matrix $\hat{A}$ in* (4.3) *has a strictly positive determinant.*

Proposition 4.2. *The matrix $\tilde{A} = \hat{A}^{-1} = \| \tilde{a}_{i,j} \|_1^n$ has the identical form as in* (4.3), *except for multiplication of each nonzero element by a positive factor and replacing p by $p + 1$* (i.e., *changing the sign of each element on the skew diagonal*).

The next assertions subsume the essential facts needed for the applicability of Theorem 3.2. We describe it in the case where $2p \leqslant n$.

Proposition 4.3. *The matrix*

$$A_1 = \|(-1)^j A_{ij}\|_{i=1,j=1}^{p\ \ n} = \left\| \begin{matrix} -1 & 0 & & \cdots & & 0 & (-1)^p c_1 \\ 0 & 1 & 0 & \cdots & 0 & (-1)^p c_2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \iddots & \iddots & \vdots \\ 0 \cdots 0 & (-1)^p & 0 \cdots 0 & (-1)^p c_p & 0 \cdots 0 \end{matrix} \right\| \tag{4.4}$$

is sign consistent of order p (SC_p) and

$$\| \tilde{a}_{n+1-j,i} \|_{i=p+1,j=1}^{n\ \ \ \ n} \quad is \quad SC_{n-p}\,. \tag{4.5}$$

The corresponding propositions for the matrices associated with the boundary forms (4.1b) will be recorded later.

It may be helpful to illustrate these propositions. Let $n = 4$, $p = 2$. Then

$$A = \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{Vmatrix}, \qquad \hat{A} = \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{Vmatrix},$$

$$A_1 = \begin{Vmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{Vmatrix}, \qquad \tilde{A} = \hat{A}^{-1} = \alpha \begin{Vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{Vmatrix},$$

where α is a positive factor. Clearly A_1 is SC_2. Observe finally that the requirement of (4.5) is satisfied, i.e.,

$$\| \tilde{a}_{n+1-j,i} \|_{i=3,j=1}^{4\ \ \ 4} = \alpha \begin{Vmatrix} 0 & +1 & +1 & 0 \\ -1 & 0 & 0 & 1 \end{Vmatrix} \quad \text{is} \quad SC_2 .$$

We turn to the proofs. Recall that n is even.

Proof of Proposition 4.1. We use induction on n. The assertion is correct for $n = 2$. Assume its validity for $n - 2$. Expand $\hat{A}$ by the first row to obtain

$$\det \hat{A} = D_1 + (-1)^{n+1}(-1)^{n+p} c_1 D_2 ,$$

where D_1 is the $n - 1 \times n - 1$ minor of the element in the first row, first column, and D_2 is the $n - 1 \times n - 1$ minor of the element in the first row, last column. Expand D_1 by the last row to get an $n - 2 \times n - 2$ determinant of the original form. Hence D_1 is strictly positive by virtue of the induction hypothesis. Expanding D_2 by the last row, we get an $n - 2 \times n - 2$ determinant of the original form multiplied by the factor $(-1)^n(-1)^{2n+p-1}$. Combining, we have $\det \hat{A} = \alpha_1 + (-1)^n \alpha_2$ where α_1 and α_2 are strictly positive. Since n is even, $\det \hat{A} > 0$ as claimed. ||

Proof of Proposition 4.2. There are three steps to the calculations.

(i) The minor of any element on the main diagonal of $\hat{A}$ is strictly positive. The evaluation is analogous to that of the previous proposition. Specifically, we employ induction on n and expand the determinants twice appropriately by rows and columns to achieve the desired result.

(ii) The minor of the element on the skew diagonal in row k has strict sign $(-1)^{n+p-k}$ and the cofactor of this element has strict sign

$$(-1)^{k+n+1-k} \cdot (-1)^{n+p-k} = (-1)^{p+k+1}.$$

The proof is the same as in (i).

(iii) The minors corresponding to the elements not located on the main or skew diagonal are zero. Consider the minor for row k, column l, $l \neq k$, $l \neq n + 1 - k$. Then all elements in rows l and $n - l$ are zero except for the entries in column $n - l$. Hence these rows are linearly dependent and the minor is zero. Therefore $\hat{A}^{-1}$ has the form as claimed.

Proof of Proposition 4.3 (the assertion of (4.4)). First, assume $p \leqslant n/2$. Any $p \times p$ submatrix Γ of A_1 will obviously have determinant zero unless the column indices of Γ consist of exactly one index from each of the pairs $(1, n)$, $(2, n - 1),\ldots, (p, n + 1 - p)$. We evaluate this determinant by expanding by rows $1, 2,\ldots, p$ successively. After $\nu - 1$ reductions of order, the resulting element in row 1, column 1 is $(-1)^\nu$ and in row 1, column $p + 1 - \nu$, of sign $(-1)^{p+1-\nu}$. Either contributes a nonzero factor of sign $(-1)^\nu$ when we expand by this row. Hence, any $p \times p$ submatrix, $p \leqslant n/2$, has a determinant of sign $(-1)^{p(p+1)/2}$.

If $p > n/2$, so that $p - 1 > n - p$, (since n is even) the matrix A_1 now has the form

$$A_1 = \left\| \begin{array}{ccccccccccc} -1 & 0 & & & & \cdots & & & 0 & & (-1)^p c_1 \\ 0 & 1 & 0 & & & \cdots & & 0 & (-1)^p c_2 & & 0 \\ & \ddots & & \ddots & & & & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \\ 0 & \cdots & 0 & & (-1)^{n/2} & & (-1)^p c_{n/2} & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & & (-1)^p c_{n/2+1} & & (-1)^{n/2+1} & & 0 & \cdots & 0 \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & \ddots & & \ddots & \\ 0 \cdots 0 & (-1)^p c_p & 0 & & & \cdots & & 0 & (-1)^p & 0 \cdots & 0 \end{array} \right\| \tag{4.6}$$

The element $(-1)^p c_p$ appears in column $n + 1 - p$, and the element $(-1)^p$ in column p. Let Γ be a $p \times p$ submatrix of A_1. Suppose the columns of index ν and $n + 1 - \nu$ are omitted for some $\nu = 1, 2,\ldots, n - p$, in forming Γ. Then, clearly, the matrix Γ displays only zeros in row ν and consequently has determinant zero. If neither column ν nor column $n + 1 - \nu$ is deleted in constructing Γ for some $\nu = 1, 2,\ldots, n - p$, then these columns are proportional (in fact, all elements except those in row ν are zero) and therefore, again, det $\Gamma = 0$. Thus, the only $p \times p$ submatrices of A_1 with possibly nonzero determinant are obtained by extracting exactly one column from each of the pairs $(1, n)$, $(2, n - 1),\ldots, (n - p, p + 1)$. Since this accounts for $n - p$ deletions, the resulting matrix will be of order $p \times p$. To evaluate this determinant we expand successively by rows $1, 2,\ldots, n - p$. As in the case $p \leqslant n/2$, the νth row contributes a nonzero factor with sign $(-1)^\nu$. Denote the remaining $2p - n \times 2p - n$ determinant by det C (C consists of rows and columns of indices $p + 2$, $p + 3,\ldots, n + 1 - p$ of A_1). After reversing the order of the columns and multiplying the matrix by $(-1)^p$, we get a matrix of the form dealt with in Proposition 4.1 and which, hence, has a nonzero determinant of fixed sign. Therefore, the assertion pertaining to (4.1) is established when $p > n/2$.

The validation of assertion (4.5) is accomplished by the same types of arguments. ||

We summarize briefly the pertinent facts concerning the boundary forms (4.1b) at the endpoint $x = 1$. The matrix for these boundary conditions is

$$B = \begin{Vmatrix} 1 & 0 & & \cdots & & 0 & & (-1)^{q+1}d_1 \\ 0 & 1 & 0 & \cdots & 0 & (-1)^{q+2}d_2 & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & \cdots & 0 & 1 & 0 \cdots 0 & (-1)^{2q}d_q & 0 \cdots & 0 \end{Vmatrix} \tag{4.7}$$

with 1's placed on the main diagonal and the entries $(-1)^{q+1}d_1$, $(-1)^{q+2}d_2$,..., $(-1)^{2q}d_q$ running down the skew diagonal. We extend B to the $n \times n$ matrix

$$\hat{B} = \begin{Vmatrix} 1 & 0 & 0 & (-1)^{q+1}d_1 \\ 0 & 1 & (-1)^{q+2}d_2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^{q+n-1} & 1 & 0 \\ (-1)^{q+n} & 0 & 0 & 1 \end{Vmatrix} \tag{4.8}$$

with 1's on the main diagonal and the elements of the skew diagonal consisting of

$$(-1)^{q+1}d_1,\ (-1)^{q+2}d_2 \ldots,\ (-1)^{2q}d_q,\ (-1)^{2q+1}, \ldots,\ (-1)^{q+n}$$

from top to bottom.

It is established as before that $\tilde{B} = \hat{B}^{-1}$ has the same form as B (apart from positive factors) except that $q + 1$ is substituted for q throughout.

Finally, the analog of Proposition 4.3 reads as

PROPOSITION 4.4. *The matrix B is SC_q and the matrix*

$$\|(-1)^j\, \tilde{b}_{n+1-j,i}\|_{i=q+1,j=1}^{n} \quad \textit{is} \quad SC_{n-q}$$

where $\tilde{B} = \hat{B}^{-1}$.

Propositions 4.1–4.4 in conjunction show that the hypotheses of Theorems 3.1 and 3.2 are satisfied for the boundary forms (4.1). Invoking the conclusions of these theorems for the case at hand produces

THEOREM 4.1. *Consider quadrature formulas of the type* (3.1) *with the boundary forms* (4.1). *Let* $\{U_i\}_1^n$ *and* $\{V_i\}_1^n$ *be the extended forms associated with the matrices* $\hat{A}$ *and* $\hat{B}$ *defined explicitly in* (4.3) *and* (4.8), *respectively. Determine the adjoint boundary forms* $\{\tilde{U}_i\}_1^n$ *and* $\{\tilde{V}_i\}_1^n$ *as in* (3.8) *and* (3.13), *respectively. The quadrature formula of type* (3.1) *best in the sense of Sard is induced by that monospline* $N_*(x)$ *satisfying the boundary conditions*

$$\begin{aligned} U_i(N_*) &= 0, \qquad i = 1, 2, \dots, p, \\ V_i(N_*) &= 0, \qquad i = 1, 2, \dots, q, \end{aligned} \tag{4.9}$$

the adjoint boundary conditions

$$\begin{aligned} \tilde{U}_i(N_*) &= 0, \qquad i = p + 1, \dots, n, \\ \tilde{V}_i(N_*) &= 0, \qquad i = q + 1, \dots, n, \end{aligned} \tag{4.10}$$

and the interpolation requirements

$$N_*(\xi_k) = 0, \qquad k = 1, 2, \dots, r. \tag{4.11}$$

A quadrature formula of type (3.1) *with boundary form* (4.1) *exists if and only if the determinant of the system* (4.9)–(4.11) *is nonzero. For* $p = q = n/2$, N_* *always exists.*

The proof of the last assertion amounts to checking that the conditions of Theorem 3.2 are indeed satisfied.

5. BEST QUADRATURE FORMULAS FOR PERIODIC BOUNDARY CONDITIONS

Let $\mathscr{P}$ denote the collection of all functions of class $C^{n-1}[0, 1]$ such that $f^{(n-1)}$ is absolutely continuous, $f^{(n)} \in L_2(0, 1)$. The space $\mathscr{P}$ is endowed with the seminorm

$$\|f\|_n = [\textstyle\int_0^1 [f^{(n)}(x)]^2 \, dx]^{1/2} \tag{5.1}$$

The family of quadrature functionals $\mathscr{Q}$ is specified to be all linear functionals of the form

$$Q_c(f) = \sum_{k=1}^{r} c_k f(\xi_k) + \sum_{i=0}^{n-1} \tilde{c}_i [f^{(i)}(1) - f^{(i)}(0)], \tag{5.2}$$

where $\{\xi_k\}_1^r$ are fixed and the subscript c indicates the dependence on the coefficient array $\{c_1, ..., c_r\}$ and $\{\tilde{c}_0, \tilde{c}_1, ..., \tilde{c}_{n-1}\}$. The functional to be approximated is, as usual, $L(f) = \int_0^1 f(x)\,dx$ and $R_c(f) = L(f) - Q_c(f)$ denotes the remainder functional. We further restrict $\mathscr{Q}$ so that $R_c(f) = 0$ for all polynomials f of degree $\leqslant n - 1$.

Our objective is to determine c_* or, equivalently, $Q_{c^*}(f)$ so that

$$\inf_c \sup_{\substack{f\in\mathscr{P} \\ \|f\|_n \leqslant 1}} | R_c(f)| = \| R_{c^*} \|. \tag{5.3}$$

We stress for the case at hand that the norm $\| R_c \|$ is evaluated with respect to functions in $\mathscr{P}$.

Each monospline $N(x)$ of degree $2n$ with knots $\{\xi_k\}_1^r$ satisfying the periodicity conditions

$$N^{(2n-i-1)}(0) = N^{(2n-i-1)}(1), \qquad i = 0, 1, ..., n - 1, \tag{5.4}$$

generates a quadrature formula of type (5.2) such that, for any f, executing a suitable integration by parts and taking account of (5.4), there results the formula

$$\int_0^1 f(x)\,dx = \sum_{k=1}^{r} c_k f(\xi_k) + \sum_{i=0}^{n-1} \tilde{c}_i [f^{(i)}(0) - f^{(i)}(1)]$$
$$+ (-1)^n \int_0^1 N^{(n)}(x) f^{(n)}(x)\,dx, \tag{5.5}$$

and we may then identify

$$R_c(f) = (-1)^n \int_0^1 N^{(n)}(x) f^{(n)}(x)\,dx.$$

The converse is also true. Every quadrature formula of the desired kind is induced by a monospline $N(x)$ of degree $2n$ satisfying (5.4). The proof of this fact paraphrases the analysis of Theorem 1.1.

The characterization of the best quadrature formula for the class $\mathscr{P}$ becomes

THEOREM 5.1. *The best quadrature formula in the sense of* (5.3) *corresponds to that monospline $N_*(x)$ of degree $2n$ with knots $\{\xi_k\}_1^r$, $r \geqslant 1$, fulfilling* (5.4) *and satisfying the orthogonality relation*

$$\int_0^1 N_*^{(n)}(x)\, S^{(n)}(x)\,dx = 0 \tag{5.6}$$

with respect to all splines $S(x)$ of degree $2n - 1$ which satisfy (5.4) *with the same knots.*

An equivalent characterization is that $N_*(x)$ is uniquely determined as the monospline of degree $2n$ satisfying the periodicity requirements

$$N_*^{(j)}(0) = N_*^{(j)}(1), \qquad j = 0, 1, \ldots, 2n - 1, \tag{5.7}$$

and the interpolation conditions

$$N_*(\xi_k) = 0, \qquad k = 1, 2, \ldots, r. \tag{5.8}$$

It is essential in Theorem 5.1 that $r \geqslant 1$ (the number of prescribed knots exceed, or equals 1).

Another version of a periodic best quadrature formula can be based on trigonometric type functions, and the interpolation theorem in Karlin and Lee [5] is relevant.

6. Extensions and Remarks

6.1. *Best L_2 Approximations*

The results of Theorems 4.1 and 5.1 can be interpreted as a characterization of best approximation in the L_n^2 norm to the zero function by monosplines satisfying appropriate boundary conditions with prescribed knots.

6.2. *General Weight Functions*

The analogous problems of determining best quadrature formulas associated with linear functionals $L(f) = \int_0^1 f(x)\, w(x)\, dx$, where $w(x)$ is a positive continuous weight function, admit the following solution: Merely replace the term $x^{2n}/(2n)!$ in the definition of a monospline by the $(2n - 1)$-fold integral

$$W_n(x) = \int_0^x \int_0^{\xi_{2n-2}} \cdots \int_0^{\xi_2} \int_0^{\xi_1} w(\xi)\, d\xi\, d\xi_1 \cdots d\xi_{2n-2}\,.$$

The correspondence between these generalized monosplines and quadrature formulas results as previously. The characterization of best quadrature is in terms of that monospline satisfying certain boundary conditions as well as vanishing at the knots.

6.3. *Tchebycheffian Quadrature Formulas*

The results summarized in this paper extend to the corresponding case of Tchebycheffian splines (for relevant definitions, see Karlin [3, Chap. 10]).

6.4. *Multi-knot Quadrature Formulas.*

A parallel analysis to that of the preceding sections enables us to characterize quadrature formulas best for $Lf = \int_0^1 f(x)\,dx$ in the sense of Sard among all quadrature formulas of the form

$$Q(f) = \sum_{j=1}^{n} B_j f^{(j-1)}(1) + \sum_{j=1}^{n} A_j f^{(j-1)}(0) + \sum_{k=1}^{r} \sum_{p=1}^{\mu_k} c_{kp} f^{(p-1)}(\xi_k) \quad (6.1)$$

which are exact for any polynomial of degree at most $n - 1$. Here we require that

$$1 \leqslant \mu_k \leqslant n. \quad (6.2)$$

The correspondence between quadrature formulas of the form (6.1), exact for polynomials of degree at most $n - 1$, and monosplines of degree n with multiple knots of the form

$$M(x) = \frac{x^n}{n!} + S_{n-1}(x), \quad (6.3)$$

where

$$S_{n-1}(x) = \sum_{k=1}^{r} \sum_{p=1}^{\mu_k} a_{kp}(x - \xi_k)_+^{n-p} + p(x) \quad (6.4)$$

and $p(x)$ is a polynomial of degree at most $n - 1$, is described by the relations

$$B_j = (-1)^{j-1} M^{(n-j)}(1), \qquad j = 1, 2, \ldots, n, \quad (6.5)$$

$$A_j = (-1)^j M^{(n-j)}(0), \qquad j = 1, 2, \ldots, n, \quad (6.6)$$

$$c_{kp} = (-1)^{p-1}[M^{(n-p)}(\xi_k-) - M^{(n-p)}(\xi_k+)], \qquad p = 1, 2, \ldots, \mu_k, \quad (6.7)$$

and remainder given by

$$R(f) = (-1)^n \int_0^1 M(x) f^{(n)}(x)\,dx. \quad (6.8)$$

In order to discern the quadrature formula of the form (6.1) best in the sense of Sard, it is convenient to replace the monospline appearing in (6.3)–(6.7) by $N^{(n)}(x) = M(x)$, where $N(x)$ is a monospline of degree $2n$ with knots of multiplicity μ_k at ξ_k, $k = 1, 2, \ldots, r$.

With this modification and letting $S(x)$ denote an arbitrary spline of degree $2n - 1$ with the same knots and multiplicities as $N(x)$, an integration by parts yields

$$(-1)^n \int_0^1 N^{(n)}(x)\, S^{(n)}(x)dx = \sum_{k=1}^{n} (-1)^k S^{(2n-k)}(x)\, N^{(k-1)}(x)\,\Big|_{x=0}^{1}$$
$$+ \sum_{l=1}^{r} \sum_{p=1}^{\mu_l} (-1)^p S^{(2n-p)}(x)\,\Big|_{x=\xi_l+}^{\xi_l-} N^{(p-1)}(\xi_l). \quad (6.9)$$

(Since $N(x)$ is a monospline of degree $2n$, and $\mu_l \leqslant n$ according to (6.2), the term $N^{(p-1)}(\xi_l)$ in (6.9) is well-defined.) The usual variational argument implies that the best quadrature formula corresponds to that monospline $N_*(x)$ which satisfies the orthogonality property

$$\int_0^1 N_*^{(n)}(x)\, S^{(n)}(x)\, dx = 0 \tag{6.10}$$

for any spline $S(x)$ of the requisite form.

The precise boundary conditions to be imposed on $N_*(x)$ will depend upon the type of the quadrature formula desired. For example, if we consider formulas of the kind

$$Q(f) = \sum_{k=1}^{r} \sum_{p=1}^{\mu_k} a_{kp} f^{(p-1)}(\xi_k), \tag{6.11}$$

then comparing (6.1) with (6.11) and referring to (6.5) and (6.6) compel that

$$N_*^{(n+j)}(0) = N_*^{(n+j)}(1) = 0, \qquad j = 0, 1, \ldots, n-1, \tag{6.12}$$

and then the orthogonality property (6.10) is satisfied for all splines of degree $2n - 1$ satisfying the boundary conditions (6.12) if and only if

$$N_*^{(p-1)}(\xi_l) = 0, \qquad l = 1, 2, \ldots, r; \qquad p = 1, 2, \ldots, \mu_l . \tag{6.13}$$

Conditions for the existence of a monospline satisfying (6.12) and (6.13) is subsumed in the content of the interpolation Theorem 2 of Karlin [4].

7. General Concepts and Prospects of Best Quadrature Formulas

It is useful and instructive to develop an abstract setting for the concept of best quadrature formula. Let L be a continuous linear functional with domain $\mathscr{B}$, a linear topological space, and let $\| \cdot \|$ be a norm or seminorm on $\mathscr{B}$.

A collection $\mathscr{Q}$ of continuous linear functionals on $\mathscr{B}$ is specified and a member $Q \in \mathscr{Q}$ is called a quadrature formula. $\mathscr{Q}$ could be but is not necessarily a linear space or a convex set. To each $Q \in \mathscr{Q}$ is associated a remainder functional

$$R_Q f = Lf - Qf. \tag{7.1}$$

A best quadrature formula is any member Q^* of $\mathscr{Q}$ satisfying

$$\inf_{Q\in\mathscr{Q}} \sup_{\substack{\|f\|\leqslant 1\\ f\in\mathscr{B}}} \| R_Q f\| = \sup_{\substack{\|f\|\leqslant 1\\ f\in\mathscr{B}}} \| R_{Q^*} f\| = \inf_{Q\in\mathscr{Q}} \| R_Q \| R_{Q^*} \|, \tag{7.2}$$

where the norm $\| R_Q \|$ is the conjugate norm with resepct to $\| \cdot \|$ (the given norm or seminorm).

We list a variety of examples.

I. Define

$$\mathscr{B} = \{f; f\in C^{n-1}[0, 1], f^{(n-1)} \text{ absolutely continuous and } f^{(n)} \in L_2[0, 1]\} \tag{7.3}$$

with the seminorm

$$\| \cdot \| = \|f\|_n = \sqrt{\int_0^1 [f^{(n)}(x)]^2\, dx}.$$

(a) Let $\{\xi_\nu\}_1^r$ $(0 < \xi_1 < \xi_2 < \cdots < \xi_r < 1)$ be fixed and let

$$\mathscr{Q} = \left\{Q : Q(f) = \sum_{i=1}^r a_i f(\xi_i),\ a_i \text{ real}\right\}. \tag{7.4}$$

Observe that (7.2) is infinite unless there exist a $Q \in \mathscr{Q}$ for which $R_Q(f) = 0$ for all polynomials of degree $\leqslant n-1$. In fact, suppose f is a polynomial of degree $\leqslant n-1$. Then

$$\| \lambda f\| = 0 \qquad \text{for all} \quad \lambda \neq 0$$

and

$$| R_Q(\lambda f)| = | \lambda | \, | R_Q(f)|.$$

So

$$\sup_{\|g\|\leqslant 1} | R_Q(g)| = \infty,$$

The above analysis shows that we can trim $\mathscr{Q}$ to include only those Q satisfying

$$Q(f) = 0 \qquad \text{for any polynomial } f \text{ of degree } \leqslant n-1.$$

It is worth noting that here $\mathscr{Q}$ is a linear space.

(b) The cases of Sections 3–5 furnish other linear spaces of quadrature formulas for the same normed spaces $\mathscr{B}$.

(c) Let

$$\mathcal{Q} = \left\{ Q;\, Q(f) = \sum_{i=1}^{r} a_i f(\xi_i) \text{ where } \{a_i\} \text{ (real) and } \{\xi_i\}_1^r, \right.$$
$$\left. 0 < \xi_1 < \xi_2 < \cdots < \xi_r < 1 \text{ are all free variables but } r \text{ is fixed} \right\}.$$

Note that in this example $\mathcal{Q}$ is not a linear space, nor is it convex, and it is not even closed. The best quadrature formula in this case, obviously, becomes the *optimal* quadrature formula as defined by Schoenberg [13].

For the specification $Lf = \int_a^b f(x)\, dx$, the identification of $R_Q(f)$ in (2.5) shows that

$$\| R_Q \| = \int_0^1 [N^{(n)}(x)]^2\, dx, \tag{7.5}$$

where N is the monospline corresponding to Q. The infimum in (7.5) taken with respect to $\mathcal{Q}$ is, therefore, equivalent to the best L_2 approximation to $x^n/n!$ by splines satisfying certain boundary conditions.

II. Consider

$\mathcal{B}$ = the class of $L_2(D)$ functions (square integrable over D with respect to two-dimensional Lebesgue measure), analytic in a domain D of the complex plane containing a real segment $[a, b]$.

Let the norm be

$$\|f\|^2 = \iint_D |f(z)|^2 \, | dz |, \tag{7.6}$$

and specify

$$L(f) = \int_a^b w(x) f(x)\, dx, \tag{7.7}$$

where $w(x)$ is a continuous function on $[a, b]$. Consider

$$\mathcal{Q} = \left\{ Q;\, Q(f) = \sum_{i=1}^{r} a_i f(\xi_i), \text{ where the free parameters are} \right.$$
$$\left. \{\xi_i\}_1^r \subset D \text{ and } \{a_i\} \text{ (complex)}, r \text{ is fixed} \right\}. \tag{7.8}$$

The space $\mathcal{B}$ has a reproducing kernel $K(z, w)$ on D such that

$$f(w) = \int_\gamma K(z, w) f(z)\, dz, \qquad w \in D,$$

where γ is a simple closed curve lying in D containing w in its interior. Then

$$Lf = \int_{\Gamma} h(z) f(z)\, dz,$$

where Γ is a simple closed curve in D surrounding $[a, b]$ and

$$h(z) = \int_a^b w(x)\, K(z, x)\, dx.$$

Therefore, if $Q \in \mathscr{Q}$, then

$$\| R_Q \| = \left\| h(z) - \sum_{i=1}^{r} a_i K(z, \xi_i) \right\|_{L_2(D)},$$

and the problem of finding a best quadrature formula for the linear functional (7.7) reduces to a best approximation problem in $L^2(D)$.

Quite generally, let the linear functional L have the representor $\hat{L}$ in the conjugate space, so that $Lf = (\hat{L}, f)$ for all $f \in \mathscr{B}$. Then the problem of finding the optimal quadrature formula, as described above, can be reduced to that of finding the best approximation to the representor $\hat{L}$ in the conjugate space by a representor of a functional in $\mathscr{Q}$; viz. that of solving

$$\inf_{\hat{Q}} \| \hat{L} - \hat{Q} \|,$$

where the norm is the conjugate norm, and $\hat{Q}$ ranges over all representors of functionals in $\mathscr{Q}$.

Some aspects of the quadrature formulas occurring in the examples of this section were investigated by several authors: see Davis [1], Eckhardt [2], Richter [7], and references therein.

III. Let $K(t, s)$ be a kernel continuous in the $L_2(d\sigma(s))$ norm with respect to parameter t, defined on $T \times T$ where $T = (-\infty, \infty)$. Suppose

$$G(\tau, t) = \int_T K(\tau, s)\, K(t, s)\, d\sigma(s), \tag{7.9}$$

where σ is a sigma finite measure on T. Note that $G(t, \tau) = G(\tau, t)$. It is easy to see that G is positive definite; in fact,

$$\sum_{i,j=1}^{m} \alpha_i \alpha_j G(t_i, t_j) = \int_T \left| \sum_{i=1}^{m} \alpha_i K(t_i, s) \right|^2 d\sigma(s) \geqslant 0,$$

for all real α_i and finite selections $\{t_i\}$ from T. Assume that

$\int_T K(t, s) f(s) d\sigma(s) \equiv 0$ with f suitably continuous implies $f \equiv 0$ (7.9a)

Let $\tilde{\mathscr{B}}_1$ be the set of all functions $\tilde{f}(t)$ of the form

$$\tilde{f}(t) = \sum_{i=1}^{m} \alpha_i G(\tau_i, t), \qquad \alpha_i, \tau_i \text{ real}; \quad m \text{ finite}, \tag{7.10}$$

endowed with the inner product (for $\tilde{g}(t) = \sum \beta_j G(s_j, t)$)

$$\langle \tilde{f}(t), \tilde{g}(t) \rangle = \sum \alpha_i \beta_j G(\tau_i, s_j),$$

which is well-defined and independent of the representations of $\tilde{f}$ and $\tilde{g}$. Then $G(t, \tau) \in \tilde{\mathscr{B}}_1$ for fixed t, and it is easy to see that $\langle G(s, \cdot), \tilde{f}(\cdot) \rangle = \tilde{f}(s)$; hence G is a reproducing kernel for $\tilde{\mathscr{B}}_1$.

Consider the mapping $U : \tilde{\mathscr{B}}_1 \to \mathscr{B}_1$, where $\tilde{f}$ in (7.10) is assigned the image

$$f(t) = \sum_{i=1}^{r} \alpha_i K(\tau_i, t)$$

and $\mathscr{B}_1$ comprises the set of all such f we find by virtue of (7.9a) that U is well defined. On $\mathscr{B}_1$ we prescribe the inner product

$$(f(\cdot), g(\cdot)) = \int_T f(t) g(t) \, d\sigma(t).$$

By virtue of (7.9) it is immediately verified that U determines an isometry of $\tilde{\mathscr{B}}_1$ onto $\mathscr{B}_1$. Let $\mathscr{B}$ and $\tilde{\mathscr{B}}$ be the Hilbert spaces obtained by completing the appropriate norms in $\mathscr{B}_1$ and $\tilde{\mathscr{B}}_1$, respectively. The mapping U can obviously be extended to map $\tilde{\mathscr{B}}$ onto $\mathscr{B}$ maintaining its isometric character. Since $\tilde{\mathscr{B}}$ is a reproducing kernel space, its elements are functions $\tilde{f}(t) = \langle G(t, \cdot), \tilde{f}(\cdot) \rangle$ and, indeed, continuous functions. Moreover if $U\tilde{f} = f$ then

$$\begin{aligned} \tilde{f}(t) &= \langle G(t, \cdot), \tilde{f}(\cdot) \rangle = (K(t, \cdot), f(\cdot)) \\ &= \int_T K(t, \tau) f(\tau) \, d\sigma(\tau). \end{aligned} \tag{7.11}$$

Consider now the problem of determining a best quadrature formula for the continuous linear functional $L(\tilde{f}) = \int_0^1 \tilde{f}(t) \, dt$ on $\tilde{\mathscr{B}}$ among the quadrature formulas of the type

$$Q(\tilde{f}) = \sum_1^r a_i \tilde{f}(t_i).$$

It is convenient to examine the equivalent problem on the isometric space $\mathscr{B}$. The induced functional $L(\tilde{f}) = LU^{-1}(f) = \bar{L}f$ in view of (7.11) takes the form

$$\int_0^1 \tilde{f}(t) \, dt = \bar{L}f = \int_T f(\tau) \, d\sigma(\tau) \int_0^1 K(t, \tau) \, dt, \tag{7.12}$$

and, correspondingly,

$$Q(\tilde{f}) = \sum_{i=1}^{r} a_i \tilde{f}(t_i) = \bar{Q}(f) = \int_T \left(\sum_{i=1}^{r} a_i K(t_i, \tau) \right) f(\tau)\, d\sigma(\tau). \quad (7.13)$$

Let

$$\mathscr{Q} = \left\{ Q;\ Q(f) = \sum_{i=1}^{r} a_i f(t_i),\ a_i \text{ real},\ \{t_i\} \in T,\ r \text{ fixed} \right\}.$$

Inspection of (7.12) and (7.13) reveals that

$$\inf_{Q \in \mathscr{Q}} \| R_Q \| = \inf_{\{a_i\},\{t_i\} \in T} \int \left| \int_0^1 K(t, \tau)\, dt - \sum_{i=1}^{r} a_i K(t_i, \tau) \right|^2 d\sigma(\tau), \quad (7.14)$$

so again the problem of optimal quadrature is one of the best L^2 approximation.

A complete characterization of the solution in (7.14), when the kernel $K(t, \tau)$ is appropriately totally positive, will be discussed elsewhere.

The example above and the problem of (7.14) have an interpretation for regression analysis of statistical time series; see Sacks and Ylvisaker [8–10] and Wahba [14].

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